field is

$$A = \iiint_V J(\vec{x}') \frac{\exp(ik_m|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d\vec{x}'.$$

From measurements in an $xy$-plane, say, to the right of the region $V$, the reconstructed field will provide the valid field at every point to the right of $V$. But the reconstructed field in an $xy$-plane that passes through $V$ is given by

$$A_R(\vec{x}) = \iiint_{V(z'<z)} J(\vec{x}') \frac{\exp(ik_m|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d\vec{x}' + \iiint_{V(z'>z)} J(x') \frac{\exp(-ik_m|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d\vec{x}'$$

which is quite different from the actual field $A(\vec{x})$.

The purpose of reconstructing the field is to construct the source distribution, which in turn provides desired information regarding the electromagnetic image of the scattering object. Even though the reconstructed field inside the scattering object is quite different from the actual field, the image obtained from the proposed technique must still represent some kind of dielectric characteristics of the scattering object. However, it is not clear exactly what characteristics it represents, and more theoretical study is needed in this regard.

Finally, we remark that active research on microwave biological imagery using a water-immersed microwave array [4] is also being performed at Walter Reed Army Institute of Research and the Johns Hopkins University Applied Physics Laboratory. Several approaches for obtaining the image from a set of limited measurements are being considered [5], [6]. Here, one of these approaches is briefly described. First, it is shown that the scattered field is equal to one that is produced by an effective charge-current distribution $(\rho_{\text{eff}}, \vec{J}_{\text{eff}})$, with

$$\rho_{\text{eff}} = -\nabla \cdot \left( \frac{\chi - \chi_m}{\chi} \vec{P} \right) \quad \text{and} \quad \vec{J}_{\text{eff}} = \frac{\chi - \chi_m}{\chi} \frac{\partial \vec{P}}{\partial t}$$

where $\chi$ and $\vec{P}$ are, respectively, the dielectric susceptibility and electric polarization inside the target, and $\chi_m$ is the dielectric susceptibility of water. Secondly, the following theorem of inverse scattering is proved: Let $J_s$ (a four-component vector) represent the charge-current distribution of the scattering target, and $A_s$ (also a 4-vector) the electric-magnetic potential of the scattered field; also, let $J_w$ be an arbitrary localized 4-vector field and $A_w$ be the 4-vector solution of the equation $(\nabla^2 + k^2) A_w(\vec{x}) = -(4\pi/c) J_w(\vec{x})$, then

$$\iiint J_w(\vec{x}) \cdot A_s(\vec{x})\, d\vec{x} = \iiint A_w(\vec{x}) \cdot J_s(\vec{x})\, d\vec{x} \qquad (3)$$

where the products are the 4-vector scalar products and the integrations are over the entire space. Equation (3) turns out to be a very useful theorem for inverse scattering problems, especially in obtaining an image of a target from limited measurements of the scattered field. If measurements on $A_s(\vec{x})$ are carried out at a set of points $\{\vec{x}_n\}$, and one takes $J_w(\vec{x}) = \sum_n J_n \delta(\vec{x} - x_n)$, then the left-hand side of (3) is completely obtainable from the limited measurements. To be more explicit, if one measures only the $y$-polarization of the scattered electric field $\vec{E}_s(\vec{x})$ at $\{\vec{x}_n\}$, then take $\vec{J}_w = \hat{y} \sum_n J_n \delta(\vec{x} - \vec{x}_n)$, so (3) gives

$$\sum_n J_n E_y(\vec{x}_n) = \frac{c}{\epsilon_m} \iiint \vec{A}_w \cdot \left[ k_m^2 \left( \frac{\chi - \chi_m}{\chi} \right) \vec{P} + \nabla \left( \nabla \cdot \frac{\chi - \chi_m}{\chi} \vec{P} \right) \right] d\vec{x} \qquad (4)$$

where $E_y(\vec{x}_n)$ is the measured $y$-component of the electric field at $\vec{x}_n$, $k_m$ the wavenumber of the microwave in water, $\epsilon_m$ the dielectric constant of water, $\vec{P}$ the polarization inside the target induced by the incident wave, $c$ the speed of light, and $\vec{A}_w(\vec{x})$ is given by $\sum_n \hat{y} J_n \exp(i|\vec{x} - \vec{x}_n|)/|\vec{x} - \vec{x}_n|$. It is possible to select a set of weights $\{J_n\}$ such that the resulting weighted field $A_w(\vec{x})$ in the region occupied by the target is very small, except for a spatially sharp peak at a focal point $\vec{x}_f$. Then the integral in the right-hand side of (4) is approximately equal to the value of

$$\frac{c}{\epsilon_m} \left[ k_m^2 \left( \frac{\chi - \chi_m}{\chi} \right) \vec{P} + \nabla \left( \nabla \cdot \frac{\chi - \chi_m}{\chi} \vec{P} \right) \right]$$

evaluated at $\vec{x}_f$, where $\vec{x}_f$ is a point inside the target.

The approach described above will require longer data processing time than the one proposed by Bolomey *et al.*, mainly because the calculation of the field $A_w(\vec{x})$ from $\{J_n\}$ is not a simple Fourier transform. It is hoped that some type of fast data processing algorithm similar to the fast Fourier transform may be developed in the near future for the purpose proposed here.

## References


[1] J. H. Bolomey, A. Izadnegahdar, L. Jofre, Ch. Pichot, G. Peronnet, and M. Solaimani, "Microwave diffraction tomography for biomedical applications," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 1998–2000, Nov. 1982.

[2] J Ch. Bolomey, L. Jofre, and G. Peronnet, "On the possible use of microwave-active imaging for remote thermal sensing," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 777–781, Sept. 1983.

[3] J. Ch Bolomey, "Fast microwave near-field recording by the modulated scattering field technique," presented at High Frequency Electromagnetic Coupling Phenomenology Meet. at Nat Bur Standards, Gaithersburg, MD, May, 1983

[4] L. E. Larsen and J. H. Jacobi, "Methods of microwave imagery for diagnostic application," in *Diagnostic Imaging in Medicine*, NATO Advanced Science Institute Series E, No. 61, C. R. Reba, ed The Hague Nijhoff Publishers, 1983, pp. 68–123

[5] T. C Guo, W. W Guo, and L E. Larsen, "Medical microwave imagery: An inverse scattering approach," in *Proc. 8th Int. Conf Infrared Millimeter Waves*, IEEE Pub No. 83CH1917-4, Miami Beach, FL, Dec. 1984.

[6] T C. Guo, W W. Guo, and L E. Larsen, "A local field study of a water-immersed microwave antenna array for medical imagery and therapy," *IEEE Trans. Microwave Theory Tech* , Special Issue on Electromagnetic Wave Interactions with Biological Systems, to be published


## Response to Reply to Comments on "Variational Methods for Nonstandard Eigenvalue Problems"

G. J. GABRIEL

In response to the author's reply to the first Comment[1] on the above paper,[2] it is affirmed that the proof of fallacy and statements contained therein stand unaltered. The author's attempt at

Manuscript received November 14, 1983.

The author is with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556.

[1] G. J. Gabriel, *IEEE Trans. Microwave Theory Tech* , vol. MTT-31, p 786, Sept 1983.

[2] I V. Lindell, *IEEE Trans Microwave Theory Tech* , vol. MTT-30, p 1194, Aug 1982.

0018-9480/84/0400-0474$01.00 ©1984 IEEE

clarification shows that his propositions are founded on contradictory suppositions and equivocation. If not for rigor and consistency, what is the excuse for using mathematics? The issues, however, are not merely ones of purity and semantics—although precision in terminology is essential to meaningful communication—but there is also an important practical consideration.

In the strict sense, in a proper variational form, eigenvalues are themselves the stationary values that the pertinent functional can take. It is this feature that makes the variational form useful sometimes as an unambiguous approximation method, requiring no foreknowledge of exact solutions to a problem. If it exists, an eigenvalue is a well-defined limit providing the user with a warning when his trial functions drift away from their mark, at least in principle. Even here, however, the method is practical only when the set of eigenvalues are not too closely spaced. In contrast, Lindell's computational procedure is ambiguous in principle because it is founded on a sufficient, but not necessary, condition. By whatever name one chooses to label the parameter $\lambda$, as quadratic roots of the expression $(f, L(\lambda)f) = 0$ they do not even guarantee that $f$ is always a solution to $L(\lambda)f = 0$, except possibly in those cases when the number zero is an eigenvalue of $L(\lambda)$ in the strict sense discussed in the Comment. Such might be the case in problems dealing with cutoff frequencies, as these correspond to vanishing wavenumbers. So long as there is ambiguity, the user is left with an uncomfortable uncertainty, unless the calculations can be compared with known exact solutions. This is self-defeating. When criteria for selection and accuracy are not given, the usefulness of a method is severely compromised. It is not surprising that Lindell's procedure should yield pleasing results sometimes, but still in others it would not. The so-called evidence is misleading because, so far, it consists of application to problems whose exact solutions are known, but no indication is given on how trial functions were selected. In any case, no computation can justify a theory founded on conceptual errors.

Turning to the author's clarifications, we observe that deleting the boundary terms in the revised form does not change the argument of fallacy. The statements concerning continuous and discrete spectra are irrelevant and inaccurate, as they confuse admissibility of a subset of functions, as solutions to a particular boundary value problem, with existence of functions in the larger parent set comprising the null space of the pertinent operator. The author falsely concludes that solutions of $L(\lambda)f = 0$ have no derivatives $\partial f/\partial\lambda$ if such solutions are admissible only for discrete values of $\lambda$. Yet, this contradicts his next supposition that $f$ changes *continuously* by $\delta f$ whenever $\lambda$ changes by $\delta\lambda$. To counter this, the function $\sin 3x$ at $\lambda = 3$ is an admissible solution to well-known boundary value problems. It certainly has the derivative $x\cos 3x$ evaluated at $\lambda = 3$. It appears that the author is confusing values of derivative with the derivative operation, and once again he hinges his argument on whether or not $(f, L'(\lambda)f)$ vanishes. In so doing, it also appears that the author is not appreciating the meaning of the zero function as an element in the range set of a mapping. In the equality $L(\lambda)f = 0$, the right member is the zero function which has zero derivatives.

The main inconsistency, however, lies in interpretations of functionals and stationarity. The expression $F(\lambda; f) = (f, L(\lambda)f)$ is a functional insofar as $f$ is concerned, but it is also a function where the parameter $\lambda$ is concerned. That is, the real or complex number $F(\lambda; f)$ to which a function $f(\underline{r})$ is mapped does depend also on the value of $\lambda$. However, the concepts of extremization of functionals and functions are significantly different. When both $f(\underline{r})$ and $\lambda$ are to be varied—$f(\underline{r})$ deviating by a *function* $\delta f(\underline{r})$ and $\lambda$ deviating by a *number* $\delta\lambda$—and given that $L(\lambda)$ is self-adjoint in the accepted sense, then the variation of $F$ is properly

$$\delta F = 2(\delta f, L(\lambda)f) + \frac{\partial F}{\partial \lambda}\delta\lambda.$$

Moreover, if for *any* reason $f$ is dependent on $\lambda$, then

$$\frac{\partial F}{\partial \lambda} = \left(\frac{\partial f}{\partial \lambda}, L(\lambda)f\right) + \left(f, L'(\lambda)f + L(\lambda)\frac{\partial f}{\partial \lambda}\right).$$

Here, the derivative $\partial F/\partial\lambda$ corresponds to $A(\lambda)$ in the first Comment, save for the fact the boundary terms are properly deleted. Obviously, if $f(\underline{r})$ is arbitrarily assumed to be independent of $\lambda$, then the only pertinent term is $(f, L'f)$, as Lindell assumes even in his revised version. The critical point, however, is the fact that $f(\underline{r})$ is *not* arbitrary, but it is constrained to be a solution of $L(\lambda)f(\underline{r}) = 0$. Even without boundary constraint, $f(\underline{r})$ becomes dependent on $\lambda$. It follows that

$$L'(\lambda)f + L(\lambda)\frac{\partial f}{\partial \lambda} = 0.$$

Consequently, when $L(\lambda)f = 0$, then $\partial F/\partial\lambda$ also vanishes, which at once implies that $\delta F$ vanishes for any nonzero $\delta\lambda$, as argued before, even if $f$ is assumed to be independent of $\lambda$. But, this is to be expected because $F(\lambda; f) = 0$ is trivially the zero eigenvalue of $L(\lambda)$ in the proper sense which is a proper stationary value of $(f, L(\lambda)f)$. The above considerations can be readily demonstrated by taking $L(\lambda) = d^2/dx^2 + \lambda^2$ on the domain $\{f(x): f(0) = 0, f'(a) = \lambda f(a)\}$. Exact solutions of this problem exist for the roots of $\tan\lambda a = 1$.

In Lindell's proposition, however, the symbol $\lambda$ is employed in two different senses. First, it is tacitly treated as a free parameter, as above, in the attempt at proving stationarity. Later, it is defined as the functional $\lambda = J(f)$, generated as roots of the expression $(f, L(\lambda)f) = 0$. So far, the author's arguments have not proven that $\delta J(f)$ actually vanishes when $L(\lambda)f = 0$. In light of the above and previous arguments, and in light of the counter example given in the first Comment, it is seriously doubtful that the roots $J(f)$ actually achieve extremal values when $L(\lambda)f = 0$. The extremal property, it is recalled, has significant bearing on practical usefulness. Nevertheless, it appears that the author missed the point of the counter example, as the boundary condition is irrelevant to the ambiguity it illustrates. In any case, the second example given above should leave no doubt.

The author has attempted to present the engineering and scientific communities with a general mathematical theory. A solitary counter example is usually sufficient to negate the sweeping generality of any assertions contained in an exposition that pretends to be rigorous and mathematical.

### *Reaction[3] to Response to Reply to Comments on "Variational Methods for Nonstandard Eigenvalue Problems" by Ismo V. Lindell[4]*

It is with a justifiable fear that the present discussion [1]–[3] does not arouse any interest in the readers of this TRANSACTIONS that I wish to react to the latest Response by Dr. Gabriel. In my view, the discussion cannot end with the Response, since it does not represent the final truth in this matter.

The main disagreement in my and Dr. Gabriel's views concerns the dependence of $f$ on $\lambda$ in the functional equation $F(\lambda; f) = 0$. Let us consider the problem in terms of four demonstrations.

1) To obtain a fresh view, let us consider the problem back-

[3] Manuscript received December 20, 1983

[4] The author is with the Helsinki University of Technology, Department of Electrical Engineering, Otakaari 5A, Espoo, 02150 Finland.

wards. Assume that $f$ and $\lambda$ only are dependent through the equation $(f, L(\lambda)f) = 0$. Hence, the following equation is valid between $\delta\lambda$ and $\delta f$:

$$2(\delta f, Lf) + \delta\lambda(f, L'(\lambda)f) = 0.$$

This means that if a solution to $F = 0$ is found such that $\delta\lambda = 0$ for any $\delta f$, we must have $Lf = 0$ with the presumptions made in [1] concerning the inner product. Hence, for the functional equation $F = 0$, if we find a solution $\lambda$ which does not change when we change the function $f$ by any small function, then $\lambda$ and $f$ must be a solution of $L(\lambda)f = 0$. There is no other hidden dependence involved in this reasoning.

2) Consider the standard eigenvalue problem, which is a special case of the present, more general formulation. The resulting functional $\lambda = (f, Lf)/(f, f)$ is proven stationary in any textbook without any additional assumption of dependence of $f$ on $\lambda$. This can also be written in an equation form as $(f, Lf) + \lambda(f, f) = 0$ or $(f, L(\lambda)f) = 0$ with $L(\lambda) = L + \lambda I$, whence the method suggested by Dr. Gabriel does not produce the normal result in this case.

3) Let us consider a similar example for functions. The equation $F(x, y) = (x-1)^2 + y^2 - 1 = 0$ describes a circle with a point at $x = 1$, $y = 1$. To study neighboring points, we set $x = 1 + \delta x$ and $y = 1 + \delta y$. Although $y = 1$ depends on $x = 1$, we do not take this dependence into account when writing the equation for the differentials: $(\delta x)^2 + 2\delta y = 0$, which shows us that $\delta y$ is of second order with respect to $\delta x$.

4) Take the example given by Dr. Gabriel, with $L(\lambda) = d^2/dx^2 + \lambda^2$ and $B(\lambda)$ defined by the two endpoint conditions $f(0) = 0, f'(a) - \lambda f(a) = 0$. This leads to the following functional equation:

$$F(\lambda; f) = \int_0^a \left(-(f')^2 + \lambda^2 f^2\right) dx + \lambda f^2(a) - 2f'(0)f(0) = 0 \quad (1)$$

which is of second degree in $\lambda$ and easily solvable. To prove the stationarity, one can set $\delta\lambda = 0$ and take a variation in $f$. After some partial integrations we readily obtain

$$2\int_0^a \delta f(f'' + \lambda^2 f)\, dx - 2\delta f(a)(f'(a) - \lambda f(a)) - 2f(0)\delta f'(0) = 0 \quad (2)$$

from which the original equations are seen to result if (2) is valid for arbitrary $\delta f$. It is no matter if we consider variation in (1) or in the solution functional $\lambda(f)$, if only we treat $\lambda$ and $f$ independent.

As a summary, it is observed that there seems to be no use in pursuing Dr. Gabriel's path through the jungle of mathematical semantics since it does not produce any useful method, whereas that given in [1] does.

## References

[1] I. V. Lindell, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 1194–1204, Aug. 1982.
[2] G. L. Gabriel, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 786–788, Sept. 1983.
[3] I. V. Lindell, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 788–789, Sept. 1983.

# Corrections to "A Planar Quasi-Optical Subharmonically Pumped Mixer Characterized by Isotropic Conversion Loss"

KARL D. STEPHAN, MEMBER, IEEE, AND TATSUO ITOH, FELLOW, IEEE

In the above paper,[1] the antenna patterns in Figs. 8 and 9 were transposed. Fig. 8 is actually the $H$-plane pattern and Fig. 9 is the $E$-plane pattern.

Manuscript received Feb. 17, 1984.

K. D. Stephan was with the University of Texas at Austin. He is now with the Department of Electrical and Computer Engineering, University of Massachusetts, Amherst, MA 01002.

T. Itoh is with the Department of Electrical Engineering, University of Texas at Austin, Austin, TX 78712

[1] K. Stephan and T. Itoh, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp 97–102, January 1984